Institut für Informatik
Norbert Blum, Marek Karpinski,
Mathias Hauptmann, Heiko Röglin

universitätbonn

Pearls of Algorithms
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Problem Set 3.2

Problem 1 (Pseudo-polynomial Time)

Find an algorithm for the knapsack problem that runs in the worst case in time $O(n^2P)$, where n is the number of items, all profits $p_1, \ldots, p_n \in \mathbb{N}$ are natural numbers, and $P = \max_{i \in \{1,\ldots,n\}} p_i$. Why does that not prove P = NP?

Problem 2 (Winner Gap)

For a given instance of the knapsack problem with profits $p_1, \ldots, p_n$, weights $w_1, \ldots, w_n$, and capacity t, we denote by x^* the optimal solution and by x^{**} the second best solution. That is,

$$x^* = \text{argmax}\{p \cdot x \mid x \in \{0,1\}^n \text{ and } w \cdot x \leq t\}$$

and

$$x^{**} = \text{argmax}\{p \cdot x \mid x \in \{0,1\}^n \text{ and } w \cdot x \leq t \text{ and } x \neq x^*\}.$$

The *winner gap* $\Delta(t)$ is defined as the margin by which the best solution is better than the second best solution:

$$\Delta = p \cdot x^* - p \cdot x^{**}.$$

Assume that $p_1, \ldots, p_n \in [0,1]$ and that $\Delta \geq n^{-k}$ for some constant k. Show that instances with these properties can be solved in worst-case polynomial time.

Hint: Round all profits to a certain accuracy, argue that this does not change the optimal solution, and apply the algorithm from Problem 1 to solve the rounded instance optimally.

Problem 3 (Continuous Isolation Lemma)

Let an instance of the knapsack problem with n items, arbitrary weights $w_1, \ldots, w_n \in \mathbb{R}_{>0}$, and an arbitrary capacity t be given. We only assume that there are at least two feasible solutions. Furthermore assume that each profit p_i is chosen independently of the other profits according to a probability density $f_i : [0,1] \to [0,\phi]$, for some $\phi \geq 1$. Show that for any $\varepsilon > 0$,

$$\mathbf{Pr}\left[\Delta \leq \varepsilon\right] \leq n\phi\varepsilon.$$

Problem 4 (Discrete Isolation Lemma)

Prove the following statement:

Let $\mathcal{F}$ be any family of subsets of a set E with n elements (i.e., $\mathcal{F}$ is a subset of the power set of E). Suppose each element $x \in E$ is independently assigned a weight $w(x)$ uniformly from the set $\{1, \ldots, N\}$, and the weight of a set $S \in \mathcal{F}$ is defined as

$$w(S) = \sum_{x \in S} w(x).$$

Then, the probability that there is a unique set in $\mathcal{F}$ of maximum weight is at least $1 - n/N$.